TEMPERATURE DISTRIBUTION IN A LIQUID LAYER ON A HORIZONTAL SOLID SURFACE

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The temperature distribution is found in an incompressible still liquid of known mass covering a horizontal solid surface at constant temperature. The exact layer thickness and the total heat content are determined. A stability condition is derived for the liquid equilibrium.

We consider a liquid layer of known mass on a solid horizontal surface whose temperature, like that of the external medium at the free surface of the liquid, is constant and also known. For a still liquid, we have the system of equations

$$\frac{d}{dz}\left(\lambda\frac{dt}{dz}\right)=0,\quad \int\limits_0^h \frac{dz}{v}=m. \tag{1}$$

At constant thermal conductivity, the liquid temperature is a linear function of the coordinate $t = t_1 + az$. Its parameters and h are to be determined.

Analogous problems have been discussed for a liquid layer covering a gravitating sphere [1] and on the inside of a rotating cylinder [2].

If the α_i are assumed constant, the boundary conditions on the temperature yield

$$a=\gamma(t_1-t_{e1}),\quad t_1+\beta x=\theta. \tag{2}$$

In particular, we have $t = t_{e2}$ when $\alpha_1 = 0$, $t = t_{e1}$ when $\alpha_2 = 0$, and $t = t_e$ when $\Delta t_e = 0$; we assume below that these cases are excluded. Corresponding to these cases we have

$$\min(t_{e1},\ t_{e2})<t,\ \theta<\max(t_{e1},\ t_{e2}),\ \operatorname{sgn} a=\operatorname{sgn}\Delta t_e.$$

We restrict the discussion to an incompressible liquid with a constant coefficient of thermal expansion:

$$v=v_0\exp(\delta t),\quad v_0=\text{const}. \tag{3}$$

After t_1 and a are eliminated with the help of (2), the integral relation in (1) becomes

$$[1-\exp(-u)]\exp(\beta u)=b(\varepsilon-u). \tag{4}$$

When $\alpha_2 = \alpha_1$, we have

$$\operatorname{sh}\frac{u}{2}=2b(\varepsilon-u);\ 2b=\gamma h_0\exp\left(\frac{t_{e1}+t_{e2}}{2}\right). \tag{5}$$

At $\delta = 0$, we easily find the final solution:

$$t_1=t_{e1}+\frac{\beta}{1+b}\Delta t_e,\quad a=\frac{\beta\gamma}{1+b}\Delta t_e,\quad h=h_0\quad (b=\beta\gamma h_0). \tag{6}$$

To find x, we must in general solve transcendental equation (4), which has a single root in the range $0 < x/\Delta t_e < 1$. All the unknown quantities are found from x:

$$t_1=t_{e1}+\beta y,\quad a=\beta\gamma y,\quad h=x/(\beta\gamma y). \tag{7}$$

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The assumed equilibrium of the still liquid is stable when $a > -g\delta t/c$ [3]; i.e., it is always stable if $\Delta t_e > 0$, or it is stable when $|a| < g\delta t_2/c$ if $\Delta t_e < 0$. Otherwise, convection may occur, generally causing a temperature distribution different from that obtained.

The total heat content of a liquid column with a base of unit area is

$$i = c\int_0^h \frac{t}{v}\,dz = \frac{cm}{\delta}\left(1 - \beta u + \frac{\beta' - \beta}{\exp u - 1}\right); \tag{8}$$

when $\delta = 0$, we have

$$i = cm\left[t_{e1} + (\beta + b/2)\frac{\Delta t_e}{1+b}\right]. \tag{9}$$

Let us consider the case of small δ ($|\varepsilon| \ll 1$) separately. Expanding the exponents in (4) in series and discarding powers of u ($|u| < |\varepsilon|$) greater than the second, we find the quadratic equation

$$(\beta - 1/2)u^2 + (1 - b)u - b\varepsilon = 0. \tag{10}$$

Its roots are real. The root which vanishes along with Δt_e is

$$x = \left[1 + \frac{\beta' - \beta}{2}\frac{b\delta}{(1+b)^2}\Delta t_e\right]\frac{b}{1+b}\Delta t_e. \tag{11}$$

We find from (7) that

$$t_1 = t_{e1} + \left[1 + \frac{\beta' - \beta}{2}\left(\frac{b}{1+b}\right)^2\varepsilon\right]\frac{\beta}{1+b}\Delta t_e,$$

$$a = \left[1 + \frac{\beta - \beta'}{2}\left(\frac{b}{1+b}\right)^2\varepsilon\right]\frac{\beta\gamma}{1+b}\Delta t_e; \tag{12}$$

$$h = h_0\exp(\delta\theta)\left(1 + \frac{\beta' - \beta}{2}\frac{b}{1+b}\varepsilon\right). \tag{13}$$

When $\delta = 0$, Eqs. (12) and (13) convert into Eqs. (6).

NOTATION

v, c, λ, δ are the specific volume, heat capacity, thermal conductivity, and coefficient of thermal expansion, respectively, of the liquid;
α is the heat-exchange coefficient (the contact thermal conductivity [1, 2, 4]);
m, i are the mass and total heat content of a liquid column with a base of unit area;
z is the coordinate, equal to 0 at the solid surface and h at the free surface of the liquid;
t, a are the absolute temperature and temperature gradient, respectively;
$\gamma = \alpha_1/\lambda$;
$\beta = 1 - \beta' = \alpha_2/(\alpha_1 + \alpha_2)$;
$h_0 = mv_0$;
$\Delta t_e = t_{e2} - t_{e1}$;
$\varepsilon = \delta\Delta t_e$;
$x = t_2 - t_1 = ah$;
$y = \Delta t_e - x$;
$u = \delta x$;
$\theta = \beta' t_{e1} + \beta t_{e2}$;
$b = \beta\gamma h_0 \exp(\delta\theta)$.

Subscripts

e, 1, 2 are the external medium, solid surface, and free surface of the liquid, respectively.

LITERATURE CITED

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